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Toric geometry and F-theory/heterotic duality in four dimensions

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Abstract: We study, as hypersurfaces in toric varieties, elliptic Calabi–Yau fourfolds for F-theory compactifications dual to $E_8 \times E_8$ heterotic strings compactified to four dimensions on elliptic Calabi–Yau threefolds with some choice of vector bundle. We describe how to read off the vector bundle data for the heterotic compactification from the toric data of the fourfold. This map allows us to construct, for example, Calabi–Yau fourfolds corresponding to three generation models with unbroken GUT groups. We also find that the geometry of the Calabi–Yau fourfold restricts the heterotic vector bundle data in a manner related to the stability of these bundles. Finally, we study Calabi–Yau fourfolds corresponding to heterotic models with fivebranes wrapping curves in the base of the Calabi–Yau threefolds. We find evidence of a topology changing extremal transition on the fourfold side which corresponds, on the heterotic side, to fivebranes wrapping different curves in the same homology class in the base.

Keywords: String Duality, Superstrings and Heterotic Strings, F-Theory.

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1. Introduction

F-Theory/Heterotic duality [1, 2, 3, 4] provides a useful way of studying nonperturbative string theory. In its original form, it states that F-theory compactified on an elliptic $K3$ surface is dual to heterotic string theory on T^2 with some choice of vector bundle V, schematically,

$$\mathrm{Het}[T^2, V] = \mathrm{F}[K3]\,. \tag{1.1}$$

In particular, the unbroken gauge group can be read off from the singularities of the elliptic fibration structure of the $K3$. Equation 1.1 is a statement about eight dimensional theories. We obtain lower dimensional versions of this duality by further compactification, and, using adiabatic arguments, applying the duality "fibrewise". For instance, in six dimensions (by further compactification of both sides of the above equation on a $\mathbb{P}^1$), we arrive at the well known duality relation

$$\mathrm{Het}[K3, V] = \mathrm{F}[\mathcal{M}_V]\,, \tag{1.2}$$

where $\mathcal{M}_V$ is an elliptic Calabi-Yau threefold which depends upon the choice of vector bundle. A very large class of such Calabi–Yau threefolds can be realised as hypersurfaces in toric varieties, and one can then establish a dictionary between the toric data and the heterotic data, including the gauge and matter content of the corresponding low dimensional effective field theories, which have $N = 1$ supersymmetry in six dimensions (see, for example, [5, 6, 7] and references therein).

If we were instead to compactify further on a (complex) surface B_2, we obtain the phenomenologically interesting duality between $N = 1$ theories in four dimensions

$$\mathrm{Het}[Z, V] = \mathrm{F}[X]\,, \tag{1.3}$$

with Z an elliptic Calabi–Yau threefold with base B_2 and X an elliptic Calabi–Yau fourfold with a three dimensional base B_3 which is a $\mathbb{P}^1$ bundle over B_2 (or a blowup thereof).

To better understand this duality, we first need a general procedure for constructing vector bundles on elliptic Calabi–Yau threefolds, and then we need to map the fourfold data to the corresponding bundle data. The first of these questions was addressed in [8, 9]. In this paper, we address the second question. Specifically, given a Calabi–Yau fourfold as a hypersurface in a toric variety, we show to read off the data necessary to construct the bundle on the heterotic side. We can then count the number of fivebranes on the heterotic side which wrap the elliptic fibre and match them to the number of threebranes on the F-theory side which is related to the Euler number of the fourfold by tadpole anomaly cancellation [10], providing the first nontrivial test of the map between toric data and bundle data. Next, we count the vector bundle moduli [8] (these have also been discussed in [11, 12, 13] and match them to the Hodge numbers (specifically, h_{31}) of the Calabi–Yau fourfold, providing the second nontrivial test of our map. Using our prescription, we will show how to construct Calabi–Yau fourfolds that yield 3 generation models with GUT groups, following [14]. We will then address the question of heterotic fivebranes wrapping curves in the base. We show that the F-theory dual of this situation consists of blowing up the corresponding curves in the fourfold base into ruled surfaces. In particular, we find that when the fivebranes wrap different curves in the heterotic base which nevertheless lie in the same homology class, the F-theory duals generally have different numbers of blowup modes, and hence different Hodge numbers. However, since the rest of the bundle data are the same, the Euler numbers are unchanged. This then raises the possibility of following a topology changing extremal transition on the fourfold in terms of degenerations of curves on the heterotic side. It is worth emphasizing here that our analysis will be purely classical. Quantum corrections will not be considered in this work.

The rest of this paper is organized as follows. In sec. 2, we summarize some relevant results in toric geometry. In sec. 3, we briefly discuss the construction of vector bundles by Friedman, Morgan and Witten [8]. In sec. 4.1, we describe our procedure for reading off the the bundle data, specifically, the η class from the polyhedra of the fourfold, and give two examples in sec. 4.2. In sec. 4.3, we find a lower bound on the η class imposed by the fourfold geometry, which is related to the stability of the corresponding vector bundle. In sec. 4.4, we discuss the construction of three generation models with GUT groups, and provide an example of such a model. In sec. 5, we study fivebranes wrapping curves in the base of the heterotic Calabi–Yau threefold, and provide an example of the topology changing transition mentioned above. sec. 6 concludes with a discussion of our findings.

During the preparation of this paper, we became aware of the work of Donagi, Lukas, Ovrut and Waldram [15] which is related to the material presented in sec. 5. They analyse the moduli space of fivebranes from an M-theory perspective, and find that when the cohomology class of the fivebranes corresponds to a reducible divisor, the moduli space of these fivebranes has several components. In our paper, we show that the F-theory duals consist of fourfolds with different Hodge numbers, but the same Euler number. Thus, the two results seem to complement to each other. After this work was complete, we received a preprint [16] which has some overlap with this work.

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2. Some results in toric geometry

In this section we briefly summarize some results in toric geometry which will be relevant to our discussion. A large class of Calabi–Yau manifolds can be realised as hypersurfaces in toric varieties, and are described, using Batyrev's construction [17, 18], by a dual pair (Δ, ∇) of reflexive polyhedra. The polyhedron Δ is called the Newton Polyhedron, and describes the monomials in the equation describing the Calabi–Yau manifold as a hypersurface in the toric variety. The dual polyhedron ∇ describes the fan of the corresponding toric variety. The Hodge numbers of the Calabi–Yau manifold are then obtained using the following formulas.

For Calabi–Yau threefolds, the only independent Hodge numbers are h_{11} and h_{12}, which are given by

$$\begin{aligned} h_{21} &= \mathrm{pts}(\Delta) \;-\sum_{\mathrm{codim}(\theta)=1}\mathrm{int}(\theta)\;+\sum_{\mathrm{codim}(\theta)=2}\mathrm{int}(\theta)\mathrm{int}(\tilde\theta)\;-\;5\,,\\ h_{11} &= \mathrm{pts}(\nabla) \;-\sum_{\mathrm{codim}(\tilde\theta)=1}\mathrm{int}(\tilde\theta)\;+\sum_{\mathrm{codim}(\tilde\theta)=2}\mathrm{int}(\tilde\theta)\mathrm{int}(\theta)\;-\;5\,, \end{aligned} \tag{2.1}$$

where $\mathrm{pts}(\Delta)$ denotes the number of integral points of Δ, $\mathrm{int}(\theta)$ stands for the number of integral points interior to a face θ and similar quantities $\mathrm{pts}(\nabla)$ and $\mathrm{int}(\tilde\theta)$ are defined for ∇. Equation 2.1 expresses the number of deformations of complex structure and Kähler classes in terms of the number of points of the polyhedra. The terms in these expressions that involve codimension-1 faces account, in the case of h_{21}, for the freedom to make redefinitions of the homogeneous variables, and in the

	s	t	u	v	x	y	w	degrees
λ	1	1	m	0	$2m+4$	$3m+6$	0	$6m+12$
μ	0	0	1	1	4	6	0	12
ν	0	0	0	0	2	3	1	6

Table 1: The scaling weights of the elliptic fibration over $\mathbb{F}_m$.

case of h_{11}, for the singularities of the toric variety which do not intersect the hypersurface. The third terms in both equations are 'correction' terms, the numbers of deformations of the corresponding hypersurface which are not visible torically. (Note that in many cases it turns out to be possible to add a certain number of points to the polyhedron under consideration so that the correction vanishes.)

Similarly, for Calabi–Yau fourfolds, the only independent Hodge numbers are h_{11}, h_{31} and h_{21}. The fourth nontrivial Hodge number h_{22} is in fact determined from

$$\chi = 48 + 6(h_{11} + h_{31} - h_{21}) = 4 + 2(h_{11} + h_{31} - 2h_{21}) + h_{22}\,, \tag{2.2}$$

which also determines the Euler number. The expressions for h_{11}, h_{31} and h_{21} are

$$\begin{aligned}
h_{31} &= \mathrm{pts}(\Delta) \; - \sum_{\mathrm{codim}(\theta)=1} \mathrm{int}(\theta) \; + \sum_{\mathrm{codim}(\theta)=2} \mathrm{int}(\theta)\mathrm{int}(\tilde\theta) \; - \; 6\,,\\
h_{11} &= \mathrm{pts}(\nabla) \; - \sum_{\mathrm{codim}(\tilde\theta)=1} \mathrm{int}(\tilde\theta) \; + \sum_{\mathrm{codim}(\tilde\theta)=2} \mathrm{int}(\tilde\theta)\mathrm{int}(\theta) \; - \; 6\,,\\
h_{21} &= \sum_{\mathrm{codim}(\tilde\theta)=3} \mathrm{int}(\tilde\theta)\mathrm{int}(\theta)\,,
\end{aligned} \tag{2.3}$$

where pts(Δ), pts(∇), θ, $\tilde\theta$, int(θ) and int($\tilde\theta$) are defined as before.

In this paper, we will mainly be interested in Calabi–Yau manifolds which are elliptic fibrations. For instance, we will consider heterotic compactifications on Calabi–Yau threefolds that are elliptically fibred over the Hirzebruch surface $\mathbb{F}_m$. Then the starting point is the hypersurface in the toric variety defined by the data displayed in table 1 [2]. Namely, start with homogeneous coordinates s, t, u, v, x, y, w, remove the loci $\{s = t = 0\}$, $\{u = v = 0\}$, $\{x = y = w = 0\}$, take the quotient by three scalings (λ, μ, ν) with the exponents shown in table 1 and restrict to the solution set of (homogeneous version of) the Weierstrass equation 2.4

$$y^2 = x^3 + f(z, z')x + g(z, z')\,, \tag{2.4}$$

where z and z' are affine coordinates on the base.

Similarly, we can construct Calabi–Yau fourfolds that are elliptically fibred over the generalized Hirzebruch surface $\mathbb{F}_{mnp}$, which is a $\mathbb{P}^1$ bundle over $\mathbb{F}_m$. The scaling weights are given in table 2.

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	q	r	s	t	u	v	x	y	w	degrees
κ	1	1	m	0	p	0	$2(m{+}p){+}4$	$3(m{+}p){+}6$	0	$6(m+p)+12$
λ	0	0	1	1	n	0	2n+4	3n+6	0	6n+12
μ	0	0	0	0	1	1	4	6	0	12
ν	0	0	0	0	0	0	2	3	1	6

Table 2: The scaling weights of the elliptic fibration over $\mathbb{F}_{mnp}$.

For the manifolds described above, the statement of the duality relation 1.3 is that F-theory compactified on a Calabi–Yau fourfold which is an elliptic fibration over $\mathbb{F}_{mnp}$ is dual to heterotic string theory compactified on a Calabi–Yau threefold which is elliptically fibred over $\mathbb{F}_m$ with a vector bundle governed by the data n and p of the fourfold. We shall, in this paper, attempt to make precise this relation between the vector bundle and the fourfold.

Toric geometry also encodes in a natural way the fibration structure (if any) of the Calabi–Yau manifolds. The authors of [19] state this for Calabi–Yau manifolds that are described by reflexive polyhedra, the integral points of the polyhedra being points in a lattice Λ. It has been shown there that in order for a Calabi–Yau n-fold to be a fibration with generic fiber a Calabi–Yau $(n-k)$-fold it is necessary and sufficient that[1]

1. There is a projection operator $\Pi\colon \Lambda \to \Lambda_{n-k}$, where Λ_{n-k} is an $n-k$ dimensional sublattice, such that $\Pi(\Delta)$ is a reflexive polyhedron in Λ_{n-k}, or
2. There is a lattice plane in $V_{\mathbb{R}}$ through the origin whose intersection with ∇ is an $n-k$ dimensional reflexive polyhedron, *i.e.* it is a slice of the polyhedron.

Conditions 1 and 2 are equivalent. If 1 or 2 hold there is also a way to see the base of the fibration torically [20]. The hyperplane H generates an $n-k$ dimensional sublattice of V. Denote this lattice V_{fiber}. Then the quotient lattice $V_{\text{base}} = V/V_{\text{fiber}}$ is the lattice in which the fan of the base lives. The fan itself can be constructed as follows. Let Π_B be a projection operator acting in V such that it projects H onto a point. Then $\Pi_B(V) = V_{\text{base}}$. When Π_B acts on ∇ the result is a k dimensional set of points in V_{base} which gives us the fan of the base if we draw rays through each point in the set.

Suppose now that we are given an elliptic Calabi–Yau threefold. The theorem of [19] tells us that in this case it is possible to find a two-dimensional hyperplane H in $V_{\mathbb{R}}$ through the origin such that its intersection with ∇ is a two-dimensional reflexive polyhedron representing the typical fiber. Let us denote it by $\nabla^{\mathcal{E}} = \nabla \cap H$. Projecting ∇ with Π_B such that $\Pi_B(\nabla^{\mathcal{E}}) = (0,0)$ yields a set of points living in a

[1] We denote, as is standard, the lattice dual to Λ (where Δ lives) by V, and its real extension by $V_{\mathbb{R}}$.

two-dimensional lattice which is what we call V_{base}. Drawing a ray from the origin $(0,0)$ through every other point gives us the fan of the base. Note that a ray may pass through more than one point and hence the number of rays, or one-dimensional cones, is generically less than the number of non-zero points in V_{base}. For elliptic Calabi–Yau fourfolds, the same picture again holds, except that the base is now a three dimensional toric variety.

In general, the elliptic fibre can degenerate over the divisors in the base. The singularity over each divisor in the base gives rise to a factor of the total gauge group. The method for reading off the singularity structure, and hence the total gauge group, was proposed in [6]. For Calabi–Yau fourfolds, there is a subtlety due to the presence of a number (generically $\frac{\chi}{24}$) of threebranes [10], required for anomaly cancellation. If the threebranes were to coincide with any of the sevenbranes wrapping the singularities, they would behave like instantons and break the observed gauge group to a smaller group [21]. Generically, however, the threebranes are located at points of the base where the elliptic fibre is smooth, and thus do not break the observed group. For the purposes of this paper, we will assume that the threebranes are indeed generic, and determine the gauge group from the singularities of the elliptic fibration structure.

3. Vector bundles on Calabi–Yau threefolds

In this section we summarize relevant aspects of the work of Friedman, Morgan and Witten [8] on vector bundles. We refer the interested reader to that work for more details.

For the purposes of this paper, we will only consider $SU(N)$ bundles and E_8 bundles, although our results should apply to other bundles as well. Friedman, Morgan and Witten construct $SU(N)$ bundles with $c_1(V) = 0$ from the spectral cover C, which is described as follows. Consider a semistable $SU(N)$ bundle on an elliptic curve E, which has a distinguished point p, the "origin". This determines a vector bundle V which splits into a sum of N line bundles, $V = \oplus_{i=1}^{N} \mathcal{N}_i$. The fact that the bundle is $SU(N)$ means that the product of the $\mathcal{N}_i$ is the trivial line bundle, and the fact that it is semistable implies that the $\mathcal{N}_i$ are all of degree zero.

For any degree zero line bundle $\mathcal{N}_i$, there is a unique point Q_i, such that $\mathcal{N}_i$ has a holomorphic section which vanishes only at Q_i and has a pole only at p. Thus V is determined by the N points Q_i on E. Since the product of the $\mathcal{N}_i$ is trivial, the sum (using addition with respect to the group law on E) of the Q_i is zero. Conversely, for any point Q_i in E, there is a unique line bundle $\mathcal{N}_i = \mathcal{O}(Q_i) \otimes \mathcal{O}(p)^{-1}$, so every N-tuple of points in E (which add up to zero) determines a semistable $SU(N)$ bundle.

Now, for an elliptic Calabi–Yau threefold , we can "fibre" the above bundle construction over the base B_2 of the elliptic fibration, obtaining an N-fold cover of the base. This is the spectral surface C of the bundle. The spectral surface is actually a section of $\mathcal{O}(\sigma)^N \otimes \mathcal{M}$, where σ is the zero section of the elliptic fibration (corresponding to a global choice of reference point p), and $\mathcal{M}$ is an arbitrary line bundle over B_2, with $c_1(\mathcal{M}) = \eta$. The class η is the single most important ingredient in the construction of the vector bundle.

Reconstructing the bundle from the spectral cover involves the Poincaré line bundle. We will not go into this topic here, but refer the reader to Ref. [8]. We simply note here a result of [8] that in general one must twist by a line bundle $\mathcal{N}$ over C in order to reconstruct a specific $SU(N)$ bundle[2]. When $H^{1,0}(C) = 0$, the classification of such line bundles on C is discrete, and $\mathcal{N}$ is uniquely determined by its first Chern class. In more general situations, we will also need to specify an element of the intermediate Jacobian $H^3(X, \mathbf{R})/H^3(X, \mathbf{Z})$, where X is the dual Calabi–Yau fourfold. However, when $h^3(X)$ is zero, which is the case for many Calabi–Yau fourfolds, this complication does not arise [8, 11]. In fact, all the examples studied in this paper satisfy $h^3(X) = 0$, although our methods will also be applicable to the more general cases.

The most general form of the line bundle $\mathcal{N}$ is [8]

$$\mathcal{N} = K_C^{1/2} \otimes K_B^{-1/2} \otimes (\mathcal{O}(\sigma)^N \otimes \mathcal{M}^{-1} \otimes \mathcal{L}^N)^\lambda\,, \tag{3.1}$$

with $c_1(\mathcal{L}) = c_1(B_2)$ (for elliptic Calabi–Yau threefolds) and suitable λ. If the square root $K_C^{1/2} \otimes K_B^{-1/2}$ does not exist, then one cannot set λ to zero, and must in fact choose λ half-integral. In fact the only circumstance in which $K_C \otimes K_B^{-1}$ has a square root is if

$$\begin{aligned} N &\equiv 0 \bmod 2\,, \\ \eta &\equiv c_1(\mathcal{L}) \bmod 2\,. \end{aligned} \tag{3.2}$$

equation 3.1 implies

$$c_1(\mathcal{N}) = \frac{1}{2}(N\sigma + \eta + c_1(B_2)) + \gamma\,, \tag{3.3}$$

with

$$\gamma = \lambda(N\sigma - \eta + Nc_1(B_2))\,. \tag{3.4}$$

Furthermore, τ-invariant bundles (where τ is the involution on the elliptic fibre) have $\gamma = 0$.

Now, the Chern classes of the corresponding $SU(N)$ bundle V can be computed to be

$$\begin{aligned} c_2(V) &= \eta\sigma - \frac{c_1(\mathcal{L})^2(N^3 - N)}{24} - \frac{N\eta(\eta - Nc_1(\mathcal{L}))}{8} - \frac{\pi_*(\gamma^2)}{2}\,, \\ c_3(V) &= 2\lambda\eta(\eta - Nc_1(B_2))\,, \end{aligned} \tag{3.5}$$

[2] A further generalization of this is mentioned in [14], but we will not consider this here.

with

$$\pi_*(\gamma^2) = -\lambda^2 N\eta\left(\eta - Nc_1(\mathcal{L})\right) . \tag{3.6}$$

The second of equations 3.5 was worked out in [14]. Note that $\frac{1}{2}c_3(V)$ is the net generation number, so we see that the only way to obtain chiral matter is to have non-τ-invariant bundles.

We do not have a spectral cover description of E_8 bundles. Semistable E_8 bundles are constructed in Ref. [8] by the method of parabolics. We will not explore the details of this construction here, but merely note that this method only yields τ-invariant bundles. Since E_8 bundles are real, the third Chern class is trivial. The second Chern class is given by

$$c_2(V) = 60(\eta\sigma - 15\eta^2 + 135\eta c_1(\mathcal{L}) - 310 c_1(\mathcal{L})^2) . \tag{3.7}$$

Before proceeding to the next section, we pause to note some important constraints on the $N = 1$ F-theory/Heterotic vacua in four dimensions. It was shown in [10] that tadpole anomaly cancellation requires that the F-Theory vacuum include $\chi(X)/24$ threebranes whose worldvolume is the uncompactified spacetime, and this requires the heterotic dual to have an equal number of fivebranes wrapping the elliptic fibre [12]. This statement is modified in the presence of the flux of the four form field strength G of the three form gauge field of eleven dimensional supergravity. Also, the location of some of the threebranes may coincide with those of the sevenbranes wrapping divisors in the base B_3 over which the elliptic fibre degenerates. These threebranes then behave like instantons, breaking the observed gauge group to a smaller group. For the purposes of this paper, we will assume that the locations of the threebranes are sufficiently generic, so that the singularities of the fibration do in fact yield the true gauge group. Then the tadpole anomaly cancellation condition is

$$\frac{\chi}{24} = n_3 + \int_X \frac{G^2}{2} . \tag{3.8}$$

It was also argued in [22] that G is quantized in half integer units. This suggests a natural relation between the four flux and the γ class, which has a similar quantization, and it was argued in Ref. [11] that in fact $\int_X \frac{G^2}{2} = -\frac{\pi_*(\gamma^2)}{2}$.

The other constraint is the general heterotic anomaly cancellation condition

$$\lambda(V_1) + \lambda(V_2) + [W] = c_2(TZ) , \tag{3.9}$$

where $\lambda(V)$ is the fundamental characteristic class of the vector bundle V (which is $c_2(V)$ for $SU(N)$ bundles and $c_2(V)/60$ for E_8 bundles), $[W]$ is the cohomology class of the fivebranes, and TZ is the tangent bundle of Z. Furthermore, for the models that we consider, $c_2(TZ) = 12c_1(B_2)\sigma + 11c_1^2(B_2) + c_2(B_2)$. Thus, we can integrate equation 3.9 over the base B_2 of the heterotic threefold, and arrive at the number of fivebranes wrapping the elliptic fibre. Thus, we arrive at a non-trivial consistency check for any map relating Calabi–Yau fourfolds and vector bundles — for any Calabi–Yau fourfold, the corresponding vector bundle will be such as to yield a number of fivebranes wrapping the elliptic fibre by equation 3.9, which must equal the number of threebranes in equation 3.8. The map that we propose in the next section yields models that do in fact satisfy this constraint.

We can also relate the bundle moduli to the Hodge numbers of the fourfold. The bundle moduli consist of even (*i.e.,* τ-invariant) and odd chiral superfields, of which there are n_e and n_o, respectively. So far, there is no known method of computing n_e and n_o, but an index theorem in [8] allows us to compute the difference $I = n_e - n_o$. From [11, 12], we can relate these to the Hodge numbers h_{21} and h_{31} of the fourfold as follows

$$\begin{aligned} h_{21} &= n_o \,, \\ h_{31} &= h_{21}(Z) + n_e + 1 = h_{21}(Z) + I + h_{21} + 1 \,, \end{aligned} \tag{3.10}$$

where unspecified Hodge numbers refer to the fourfold X. The index I is given by [8]

$$I = -\frac{1}{2}\sum_{i=0}^{3}(-1)^i \mathrm{Tr}_{H^i(Z,Ad(V))}\tau \,. \tag{3.11}$$

For τ-invariant bundles, we get

$$I = r - 4\int_\sigma \lambda(V)|_\sigma - 3\int_Z c_1(\mathcal{L})\lambda(V) \,, \tag{3.12}$$

where r is the rank of the structure group of the bundle and $\sigma|_\sigma = -c_1(\mathcal{L})|_\sigma$. This formula cannot be applied for bundles that are not τ-invariant. For non-τ-invariant $SU(N)$ bundles, we compute I using another formula of [8]

$$I = -1 + \int_B e^\eta (1 + e^{-2c_1(\mathcal{L})} + e^{-3c_1(\mathcal{L})} + \cdots + e^{-Nc_1(\mathcal{L})})\mathrm{Td}(B) \,, \tag{3.13}$$

where Td is the Todd class, defined for any complex manifold W by

$$\mathrm{Td}(W) = 1 + \frac{c_1(W)}{2} + \frac{c_2(W) + c_1^2(W)}{12} + \cdots \tag{3.14}$$

In fact, it was shown in [8] that equation 3.13 agrees with equation 3.12 for τ-invariant bundles. Using these formulas, we obtain our second consistency check of the map that we propose in the next section, and all the models that we study satisfy this constraint.

We have not yet discussed the third independent Hodge number of the fourfold, namely, h_{11}. A formula in [12] gives

$$h_{11} = h_{11}(Z) + 1 + \mathrm{rank}(G) \,, \tag{3.15}$$

where rank(G) is the rank of the unbroken non-abelian gauge group. However, when we have fivebranes wrapping curves in the heterotic base B_2 as in sec. 5, this formula will have to be modified to include the number of blowups of the base B_3. The correct formula (when Z has a smooth Weierstrass fibration, which will be true of all the models we study in this paper) by analogy with the six dimensional situation, is

$$h_{11}(X) = 1 + h_{11}(B_3) + \mathrm{rank}(G)\,, \tag{3.16}$$

where G is the unbroken gauge group.

4. Mapping toric data to vector bundle data

4.1 The general technique

Recall that we can consider the Calabi–Yau fourfold to be a $K3$ fibration over B_2. The $K3$ fibre itself has an elliptic fibration compatible with the elliptic fibration structure of the Calabi–Yau fourfold. In general, the elliptic fibre can degenerate over several divisors in the threefold base of the Calabi–Yau fourfold, leading to enhanced gauge symmetry. For the purposes of this paper, we will only consider the situation when these singularities lie in the $K3$ fibre. This is the analogue in four dimensions of the six dimensional case when the gauge group was purely perturbative (from a heterotic perspective), *i.e.,* a subgroup of the heterotic $E_8 \times E_8$ gauge group. The unbroken gauge symmetry was then the commutant of the structure group of the vector bundle, and was related to the singularity type by identifying a singularity of type sec. ADE with the corresponding **ADE** group. Using the adiabatic argument, therefore, we conclude that in the fourfold situation (if all the singularities of the elliptic fibration lie in the $K3$ fibre), the gauge group that we read off from the singularities is just the commutant in $E_8 \times E_8$ of the structure group of the vector bundle. The assumption that all the singularities of the elliptic fibration lie in the $K3$ fibre means in particular that the Calabi–Yau threefold on the heterotic side has a *smooth* Weierstrass fibration.

We still need to specify the bundles themselves. For this, we need to specify, among other things, the η and γ classes. We relate them to the toric data as follows. The base of the $K3$ fibre is a $\mathbb{P}^1$ which is precisely the $\mathbb{P}^1$ fibre of B_3 over B_2. From the discussion in sec. 2 (since the $K3$ fibration of the Calabi–Yau fourfold is compatible with its elliptic fibration), the fan of this $\mathbb{P}^1$ is seen as a slice through the origin in the fan of B_3. Now the fan of $\mathbb{P}^1$ consists simply of two rays, R_1 and R_2, opposite each other. These correspond to divisors in the base B_3 (see table 3), and the singularities *sec.*G_1 and *sec.*G_2 of the elliptic fibration over R_1 and R_2 (which are read off from the preimages of R_1 and R_2 under the map which projects the polyhedron ∇ of the Calabi–Yau fourfold onto the fan of the base) give rise to the gauge group $G_1 \times G_2$ which is the commutant in $E_8 \times E_8$ of the structure group

$V_1 \times V_2$ of the vector bundle. Thus V_1 and V_2 are naturally associated to the rays R_1 and R_2 in the fan of B_3.

For each R_i, we define a divisor t_i in B_2 as follows. Due to the linear relations of the fan [23], we have $R_2 = R_1 + \sum A_j D_j$ where the sum runs over all the other divisors D_j in B_3 and the A_j are integers. Now, $R_1.R_2 = 0$, so $R_1.(R_1 + \sum A_j D_j) = 0$. Clearly, in this expression, we can restrict the sum to the divisors D_j that actually intersect R_1. If $\pi(D_j)$ is the image of D_j under the projection π from B_3 to B_2, then we write

$$t_1 = \sum_{D_j.R_1 \neq 0} A_j \pi(D_j)\,. \tag{4.1}$$

(For experts in toric geometry, t_1 is a linear combination of divisors in Star(R_1). A similar expression, this time as a linear combination of divisors in Star(R_2), then holds for t_2.)

We now define $\eta(V_i)$ for the vector bundles V_i as

$$\eta(V_i) = 6c_1(B_2) - t_i, (i = 1, 2)\,. \tag{4.2}$$

We claim that this definition of η gives us precisely the η of Friedman, Morgan and Witten [8] where η was defined in terms of a class t that satisfied $r(r+t) = 0$ for the class r of the zero section of B_3 over B_2. In our construction, we identify R_1 with r, and then our definition of t_1 matches that of t. When B_3 is a $\mathbb{P}^1$ bundle over B_2 (and not a blowup thereof), then t_2 is simply $-t_1$, and our definitions reproduce the definitions in [8]. However, our definitions generalize naturally to the case when B_3 is a blowup of a $\mathbb{P}^1$ bundle over B_2 which will become important in sec. 5.

Since the definitions of t and η above are rather abstract, we illustrate them with the following example. Consider the situation when the heterotic Calabi–Yau threefold is elliptically fibred over $\mathbb{F}_m$, while the dual Calabi–Yau fourfold is fibred over $\mathbb{F}_{mnp}$. The fan of $\mathbb{F}_{mnp}$ is generated by rays through the points in table 3, where we have also labeled the corresponding divisors.

Divisors	Points
R_1	$(0,0,1)$
D_1	$(1,m,p)$
D_2	$(0,1,n)$
D_3	$(0,-1,0)$
D_4	$(-1,0,0)$
R_2	$(0,0,-1)$

Table 3: The points generating the fan of $\mathbb{F}_{mnp}$.

Note that the rays R_1 and R_2 form a slice $(0,0,z)$ of the fan through the origin, and give the $\mathbb{P}^1$ fibre of $\mathbb{F}_{mnp}$, while projecting out the R_i, *i.e.,* mapping (x,y,z) to (x,y) gives the fan of the base $\mathbb{F}_m$. The linear relations of the fan [23] imply that

$$\begin{aligned} D_1 &= D_4\,, \\ D_3 &= D_2 + mD_1\,, \\ R_2 &= R_1 + pD_1 + nD_2\,, \end{aligned} \tag{4.3}$$

The first two of these imply that $D_1 = D_4 = f$, the fibre class, $D_2 = C_0$, the zero section and $D_3 = C_\infty$, the infinity section of $\mathbb{F}_m$ regarded as a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ (strictly speaking, we should take $\pi(D_i)$, where π is the projection from $\mathbb{F}_{mnp}$ to $\mathbb{F}_m$). Furthermore, the rays R_1 and R_2 generate the fan of the $\mathbb{P}^1$ fibre of $\mathbb{F}_{mnp}$ over the base $\mathbb{F}_m$. Following the prescription given above, we read off $t_1 = pf + nC_0 = -t_2$, and thus obtain η_1 and η_2, which agrees with the results of [9].

Now, for $SU(N)$ bundles, γ, the analogue of the four flux, is not determined by the polyhedron, since we have to specify the four flux in addition to specifying the fourfold. However, because of tadpole anomaly cancellation 3.8, and because the number of fivebranes wrapping the elliptic fiber on the heterotic side is non-negative, we find that the γ class cannot be arbitrary, but is often restricted to a small set of possibilities. Thus, $\frac{1}{2}c_3(V)$, the net number of generations, which by a formula of Curio [14] is related to the γ class, is also restricted to a small number of possibilities for any given η. Thus, we find that we must tune η to very special values if we wish to obtain, say, a model with 3 generations. Note that the γ class is well defined only for bundles which can be described by the spectral cover method, including $SU(N)$ bundles. For E_8 bundles, the γ class does not exist[3]. We now give a couple of illustrative examples.

4.2 Two examples

Consider, for example, the fourfold which is the elliptic fibration over $\mathbb{F}_{mnp}$ with $m = 1, n = 12$ and $p = 18$. This has an E_8 singularity over the zero section (*i.e.,* over the divisor which we have called R_1). The Hodge numbers are[4] $h_{11} = 12, h_{31} = 27548, h_{21} = 0, h_{22} = 110284$ and Euler number $\chi = 165408$. Note that $h_{11} = 4 + \mathrm{rank}(E_8)$, which is the analog of a similar relation in threefolds, which agrees with equation 3.16. Here, and in what follows, we will always use the notation of table 3 to describe divisors in the base.

The heterotic Calabi–Yau threefold is an elliptic fibration over $\mathbb{F}_1$. This has a smooth Weierstrass model, and Hodge numbers $h_{11} = 3, h_{21} = 243$. From the fourfold, we see that the structure group V_1 is trivial, while $V_2 = E_8$. In this case, since we do not have any SU bundles, we have no bundle analogue for the four flux, hence, we set the total four flux on the F-theory side to zero. Clearly, the only possible value for $\eta(V_1)$ is 0. (A trivial bundle with non-trivial η is absurd). Therefore, we must have $t_1 = -6c_1(\mathbb{F}_1)$. This is actually the case — if C_0 and f denote the zero section and fibre class of $\mathbb{F}_1$, we have $c_1(\mathbb{F}_1) = 2C_0 + 3f$, while t_1 from the fan of $\mathbb{F}_{mnp}$ is easily seen to be $-12C_0 - 18f$. Also, in our example, $\eta(V_2) = 12c_1(\mathbb{F}_1)$. Given this data, and knowing that $C_0^2 = -1, f^2 = 0, C_0.f = 1$, we

[3] I am grateful to E. Witten for explaining this point.

[4] The Hodge numbers of all the manifolds discussed in this paper have been computed using the program `POLYHEDRON`, written by P. Candelas.

can readily compute $\int_{B_2} \lambda(V)$ to be -6800. The number of fivebranes, n_5, is

$$n_5 = \int_{B_2} c_2(TZ) - \int_{B_2} \lambda(V) = 92 + 6800 = 6892\,, \tag{4.4}$$

which is precisely $\frac{\chi}{24}$, as would be expected from $n_5 = n_3$ and $\frac{1}{2}G^2 = 0$. Note that the net number of generations is zero, since we have a τ-invariant bundle.

The bundle moduli can be computed using equations 3.7 and 3.12. With $\eta = 12c_1(\mathbb{F}_1)$, we find, after some trivial algebra, that $I = 27304$. Also we have $h_{21}(Z) = 243$, so that we obtain $1 + h_{21}(Z) + I + h_{21}(X) = 27548 = h_{31}(X)$, which satisfies the second consistency check of our map.

In studying this model, we have obtained an important condition. This is that if we have a divisor corresponding to unbroken E_8, the corresponding t *must* be equal to $-6c_1(B_2)$, so that η vanishes. This is the fourfold analogue of the situation for threefolds where the self-intersection of any divisor corresponding to unbroken E_8 had to be -12, and vice versa. Similarly, we conclude that if $t = -6c_1(B_2)$, then the elliptic fibration *must* have an $sec.E_8$ singularity along the corresponding divisor, leading to an unbroken E_8 gauge group. This is because the corresponding bundle has trivial η, and hence must be trivial. We will have more to say about this shortly.

Now enforce an E_6 singularity along the ray R_2, which lies opposite the ray R_1, by adding points to ∇, as in Ref [5]. The fourfold has Hodge numbers $h_{11} = 18, h_{31} = 1670, h_{21} = 0, h_{22} = 6796$ and Euler number $\chi = 10176$. Note that h_{11} agrees with equation 3.16.

On the heterotic side, we again have an elliptic fibration over $\mathbb{F}_1$, but we now have an $SU(3)$ bundle with $\eta = 12c_1(\mathbb{F}_1)$ and we are forced to have a non-trivial γ class from equation 3.2. We then have $\int_{\mathbb{F}_1} c_2(V) = -332 + \lambda^2(1296)$, with $\lambda = k + \frac{1}{2}$ for some integer k. This gives $n_5 = 424 - 1296\lambda^2$, and requiring that n_5 be non-negative yields $\lambda = \pm\frac{1}{2}$, giving $n_5 = n_3 = 100$ and $\frac{1}{2}G^2 = 324$. Finally, $\frac{\chi}{24} = 424 = n_3 + \frac{1}{2}G^2$, as expected. The net number of generations is then $\frac{1}{2}c_3(V) = \pm 432$, corresponding to $\lambda = \pm\frac{1}{2}$. Note that in this case, the net number of generations is restricted to two possible values, corresponding to the two possible values of the γ class. This illustrates the statements made above, that the γ class, though not specified by the choice of fourfold, is nevertheless restricted to a small set of values by the condition that the number of fivebranes be positive.

We can now compute the bundle moduli using equations 3.5 and 3.13. After some trivial algebra, we find $I = 1426$, and so $1+h_{21}(Z)+I+h_{21}(X) = 1+243+1426+0 = 1670 = h_{31}(X)$, as expected. Note that because our $SU(3)$ bundle is not τ-invariant, we cannot use equation 3.12 to compute I. In fact, equation 3.12 gives $I = 130$, which violates our consistency check. This, of course, is not a problem, since equation 3.12 was derived for τ-invariant bundles and is not expected to hold for bundles which are not τ-invariant.

4.3 A lower bound for η

While studying the first example above, we obtained the condition that $\eta = 0$ must correspond to an E_8 singularity. We also mentioned that this is the analogue of the situation in six dimensions, where an E_8 singularity implies that the corresponding divisor in the base has self-intersection -12. Now, in six dimensions, for any gauge bundle H on the heterotic side, we also had to have a minimum number of instantons (*e.g.,* we require a minimum of 4 instantons for a $SU(2)$ bundle, and 10 instantons for an E_8 bundle). Since η is the four dimensional analogue of the instanton number, it is natural to wonder if there is a "minimum" η for any gauge bundle. Since η is a divisor, we need to define the notion of "minimum". We define the "minimum value" $\eta_{\text{min}}(H)$ for any gauge bundle with structure group H to be such that for any η with $\eta_{\text{min}}(H) - \eta$ an effective divisor, the singularity corresponding to (*i.e.,* enforced by) η is worse than G, where the corresponding group G is the commutant of H in E_8. In particular, if G is a subgroup of E_8, then we see that $\eta_{\text{min}}(H)$ must itself be an effective divisor, since $\eta = 0$ enforces an E_8 singularity, which is worse than G. So we have reason to believe that the notion of η is well defined.

Another motivation for the existence of a lower bound on η is provided by the toric data. When we compactify F-theory on an elliptic Calabi–Yau threefold with base $\mathbb{F}_m$, reflexivity of the polyhedron ∇ forces us to add points to $\Pi^{-1}(C_0)$ (where Π is the projection from ∇ to the base $\mathbb{F}_m$), and the greater the value of m, the more points we need to add. These points signal a degeneration of the elliptic fibre over C_0. The greater the value of m, the worse the singularity. Now, in the case of fourfolds, the class t plays much the same role with respect to R_1 as m does for C_0. Thus, the "bigger" the class t, the worse the singularity forced over R_1, and the smaller the value of η. So, for any given η, there is a "minimum" singularity forced over R_1, and hence a "minimum" gauge group. In other words, to any given vector bundle, there must correspond a minimum value of η.

We will propose an expression for the lower bound for η by analogy with the six dimensional situation. The argument given below is very similar to the one in sec.6.4 of ref. [24]. Consider F-theory compactified on a Calabi–Yau threefold which is an elliptic fibration over $\mathbb{F}_m$. Consider the zero section C_0 of $\mathbb{F}_m$, and let the elliptic fibre degenerate over C_0, giving a G singularity. Then the discriminant locus $\Delta = 12c_1(\mathbb{F}_m)$ vanishes to order $\delta(G)$ over C_0. The values of $\delta(G)$ for any gauge singularity are obtained from Tate's algorithm (see, for example, ref. [3]). If we subtract this contribution from the discriminant, we expect the remainder $\Delta' = \Delta - \delta(G)C_0$ to have only transverse intersections with C_0, so that $\Delta'.C_0 \geq 0$ (otherwise the singularity over C_0 would be worse than G). Thus we obtain, since $c_1(\mathbb{F}_m) = 2C_0 + (m+2)f$, that $m \leq \frac{24}{12-\delta(G)}$. Now, the number of instantons in the corresponding bundle is $12 - m$, so we see that the minimum number of instantons to enforce a G singularity is $12 - \frac{24}{12-\delta(G)}$.

For example, the number of instantons that will cause the degeneration of the fibre to be no worse than I_1 (*i.e.,* no singularity) is $\geq 12 - 2\frac{2}{5}$ (since $\delta(I_1) = 2$), so we need at least 10 instantons for a smooth fibre, *i.e.,* we need at least 10 instantons for an E_8 bundle. Similarly, the number of instantons required to enforce an $SU(2)$ singularity is $\geq 12 - 2\frac{2}{3}$ (since $\delta(SU(2)) = 3$), so we must have at least 10 instantons for an $SU(2)$ singularity (with 9 instantons, we get $SU(3)$). Turning this around, we can then say that we must have at least 10 instantons to fill out an E_7 bundle. Similarly, since $\delta(E_7) = 9$, the minimum number of instantons for an E_7 singularity is $12 - 8 = 4$, *i.e.,* we must have at least 4 instantons for an $SU(2)$ bundle.

We could attempt to derive a lower bound for η in the fourfold case in the same way. For instance, we could analyse the degeneration of the fibre over R_1. However, we run into difficulty here since $\Delta'.R_1$ is not a number (since the base of the elliptic fibration is now a threefold). We can, however, approach the problem differently. We note that the quantity $\eta = 6c_1(B_2) - t$, where B_2 is the heterotic base, is analogous to the instanton number $12 - m$ [8]. Basically, we replace the instanton number k by the class $\frac{k}{2}c_1(B_2)$. So we propose the following ansatz: the minimum η for any singularity G is

$$\eta_{\text{min}}(H) = \left(6 - \frac{12}{12 - \delta(G)}\right) c_1(B_2)\,, \tag{4.5}$$

where H is again the commutant of G in E_8. For example, if $G = E_8$, since $\delta(E_8) = 10$, we find that $\eta_{\text{min}}(E_8) = 0$, which is consistent with our previous results. It will be interesting to see an actual derivation of this condition directly, without appealing to the analogy with threefolds.

It is extremely interesting to explore the meaning of this lower bound for η. It turns out that it is related to the stability of the corresponding vector bundle. The following argument was explained to me by R. Friedman. From Corollary 6.3 of [25] it follows that at least one of the line bundles $\mathcal{M}, \mathcal{M}\otimes\mathcal{L}^{-2}, \ldots, \mathcal{M}\otimes\mathcal{L}^{-n}$ must have a section, and since $\mathcal{L}^4$ or $\mathcal{L}^6$ has a section, this suggests $\eta = c_1(\mathcal{M})$ must have a "lower bound". (Recall that $c_1(\mathcal{L}) = c_1(B_2)$). Furthermore, if $\mathcal{M}$ is trivial, then sec. 5.6 of Ref. [25] states that the vector bundle is usually never stable, so that $\mathcal{M}$ is usually at least as effective as $\mathcal{L}^2$.

In fact, we see that equation 4.5 agrees with the above statement. For an $SU(2)$ bundle, corresponding to an E_7 singularity, we see that $\eta_{\text{min}}(SU(2)) = 2c_1(B_2)$, corresponding to a line bundle $\mathcal{M}_{\text{min}} = \mathcal{L}^2$. Since $SU(2)$ is essentially the "smallest" vector bundle we can have, this means that in general, $\mathcal{M}$ is at least as effective as $\mathcal{L}^2$, as stated above. But equation 4.5 was obtained from purely geometric considerations on the Calabi–Yau fourfold. Thus, we see one of the miracles of string duality - the elliptic Calabi–Yau fourfold "knows" about the stability of vector bundles on the Calabi–Yau threefold!

Before we leave this topic, we mention in passing that η is also bounded from above. For supersymmetric vacua, we need the class of fivebranes to be effective (otherwise we would have antibranes [15]). Then it follows trivially that $\sum_i \eta(V_i) \leq 12c_1(B_2)$, where we sum over all the bundle factors.

4.4 Three generation models

Given the relation between bundle data and fourfold data, we are in a position to attempt to construct models with three generations and GUT groups. Consider, for instance, a three generation model with E_6 gauge symmetry. This corresponds to an $SU(3)$ bundle with $\frac{1}{2}c_3(V) = 3$. This puts constraints on the bundle data η and $\gamma = \lambda\eta(\eta - 3c_1(B_2))$, and one can attempt to find solutions to these constraints.

Some of these solutions are given in [14]. For $SU(3)$ bundles on Calabi–Yau threefolds with base $\mathbb{F}_m$, the solution given there is $\eta = f$, $\lambda = -\frac{1}{2}$. (Note that $m \leq 2$ for the heterotic Calabi–Yau threefold to have a smooth Weierstrass model.) This solution, is however, impossible, by the lower bound of the previous section. Since $\delta(E_6) = 8$, we have $\eta_{\text{min}}(SU(3)) = 3c_1(B_2) = 6C_0 + 3(m+2)f > f$, so this solution violates the lower bound. In fact, this solution also violates the weaker condition that $\mathcal{M}$ should be at least as effective as $\mathcal{L}^2$. Indeed, it is impossible to construct a reflexive polyhedron with this value of η and an E_6 singularity — the singularity forced over R_1 is worse than E_6.

While it is possible to attempt to work out all the solutions to the above constraints, we will present just one. The elliptic fibration over $\mathbb{F}_{1,6,8}$ with an E_6 singularity imposed over R_1 and with four flux fixed by setting $\lambda = \frac{1}{2}$ is dual to a heterotic compactification on the smooth Calabi–Yau threefold over $\mathbb{F}_1$, with an $SU(3)$ bundle with $\eta = 6C_0 + 10f$ and γ determined by $\lambda = \frac{1}{2}$, as well as a τ-invariant E_8 bundle with $\eta = 18C_0 + 26f$. The polyhedron ∇ consists of the points shown below:

```
(-1,0,0,2,3), (0,-1,0,2,3), (0,0,0,-1,0), (0,0,0,0,-1),
(0,0,0,0,0), (0,0,1,0,0), (0,0,0,0,1), (0,0,1,0,1),
(0,0,0,1,1), (0,0,1,1,1), (0,0,2,1,1), (0,0,0,1,2),
(0,0,1,1,2), (0,0,2,1,2), (0,0,-1,2,3), (0,0,0,2,3),
(0,0,1,2,3), (0,0,2,2,3), (0,0,3,2,3), (0,1,6,2,3), (1,1,8,2,3)
```

The Calabi–Yau fourfold has Hodge numbers $h_{11} = 10, h_{31} = 9231, h_{21} = 0, h_{22} = 37008$ and Euler number $\chi = 55494$. Note that $h_{11} = 4 + \text{rank}(E_6)$. Note also that since $h_{21} = 0$, the spectral curve of the $SU(3)$ bundle has $h_{10} = 0$, so that the bundle is completely specified by the data given above. In particular, we find $n_5 = n_3 = 2310$, $\frac{1}{2}G^2 = -\frac{\pi_*(\gamma^2)}{2} = 9/4$, and $\chi = 24(n_3 + \frac{1}{2}G^2)$, as expected. Finally, this is a 3 generation model: $\frac{1}{2}c_3(SU(3)) = 3$. (For computing the net generation number, we ignore the contribution from the hidden sector bundle, which in our example gives zero anyway.)

Next, let us count the moduli of our bundles. Using equations 3.7 and 3.12, we find for the τ-invariant E_8 bundle, $I_{E_8} = 8918$. Also from equation 3.13, we find, for the non-τ-invariant $SU(3)$ bundle, $I_{SU(3)} = 69$, so that $1 + h_{21}(Z) + I_{E_8} + I_{SU(3)} + h_{21}(X) = 1 + 243 + 8918 + 69 + 0 = 9231 = h_{31}(X)$, consistent with our expectations. As in example *4.2.2*, note that we cannot use equation 3.12 to compute I for the $SU(3)$ bundle since it is not τ-invariant. In fact, equation 3.12 predicts $I_{SU(3)} = 60$, which is wrong, but this is not a problem since equation 3.12 is only valid for τ-invariant bundles anyway.

In our example, the hidden sector group was completely broken, which may not be phenomenologically desirable. However, we could also consider models with unbroken hidden sector groups, by adding points to ∇ over the ray R_2, as in the six dimensional situation [5]. This leads to a large number of possibilities for models with three generations. Since we have more than one choice for η and γ yielding three generation models to begin with, we see that we can construct many Calabi–Yau fourfolds that yield three generation models. Note also that while we have only done the analysis for E_6, a similar analysis can also be done for the GUT groups $SO(10)$ and $SU(5)$.

In passing, we note that the methods of Klemm et. [26] can be used to identify all the divisors with arithmetic genus 1, *i.e.,* those that can contribute to the superpotential. We list those divisors below:

```
(0,0,0,2,3), (0,0,1,0,0), (0,0,1,0,1), (0,0,1,2,3),
(0,0,2,1,1), (0,0,2,1,2), (0,0,2,2,3), (0,0,3,2,3), (0,1,6,2,3)
```

Note that the first of these is a horizontal divisor, and so does not contribute to the superpotential in F-theory.

5. Fivebranes and Extremal Transitions

In the work of Friedman, Morgan and Witten [8], as well as in all the examples studied so far, one only considers models with $\eta(V_1) + \eta(V_2) = 12c_1(B_2)$. Among other things, this ensures that all the fivebranes on the heterotic side are wrapping the elliptic fibre of the Calabi–Yau threefold. We can now ask what happens when the $\eta(V_i)$ do not add up to $12c_1(B_2)$, say, $12c_1(B_2) - \eta(V_1) - \eta(V_2) = C$, where C is a divisor in B_2. By equation 3.9, the cohomology class of the fivebranes includes the class $C\sigma$. One necessary constraint on the class C is that it must correspond to an effective divisor, a condition obtained in [15]. The authors of that work also study the moduli space of these fivebranes. Here, we will discuss the F-theory picture dual to this situation.

The natural interpretation of these fivebranes (which was also proposed by [15]) is that they wrap holomorphic curves in the base whose class is precisely C. For the purposes of this paper, we will only consider the situation when the curve C is a $\mathbb{P}^1$. We now propose a heuristic argument for guessing the F-theory dual to this picture. Later, we will construct models that support our argument. In the absence of fivebranes wrapping curves in B_2, the base of the Calabi–Yau fourfold on the F-theory side is a $\mathbb{P}^1$ fibred over B_2. Thus for any $\mathbb{P}^1$ in the heterotic B_2, we have a corresponding $\mathbb{P}^1$ in the F-theory B_3. Consider the limit when this $\mathbb{P}^1$ becomes large, but the rest of the Calabi–Yau fourfold remains small. In this limit, we arrive at something that looks like a six dimensional situation. Wrapping a fivebrane on the large $\mathbb{P}^1$ then looks like turning on a fivebrane in the six dimensional sense, which corresponds to a blow up of the F-theory base in six dimensions. Now, the F-theory base B_3 can be regarded as being fibred over the wrapped $\mathbb{P}^1$, so we are really blowing up over this wrapped $\mathbb{P}^1$. Thus we arrive at our F-theory dual: wrapping a fivebrane over a $\mathbb{P}^1$ in the heterotic base corresponds to blowing up the corresponding $\mathbb{P}^1$ in B_3 into a ruled surface (a $\mathbb{P}^1$ fibred over the original $\mathbb{P}^1$). This is therefore a "fibred" version of the six dimensional story when we have extra tensor multiplets. Carrying this analogy further, we note that the extremal transition on the F-theory side corresponding to the appearance of an extra fivebrane (in six dimensions) was described in [3]. In that work, the authors show that the appearance of an extra fivebrane corresponds to replacing a singular point by the del Pezzo surface dP_8. Similarly, one can argue that in the four dimensional case, wrapping a fivebrane around a $\mathbb{P}^1$ in B_2 corresponds to replacing the corresponding (singular) curve in the Calabi–Yau fourfold by a dP_8 fibred over it.

5.1 Example 1

Let us consider some specific examples to illustrate this idea. Consider F-theory compactified on the Calabi–Yau fourfold which is elliptically fibred over $\mathbb{F}_{100}$ (we refer the reader table 3 and equation 4.3, where now $n = p = 0$). The heterotic dual consists of an $E_8 \times E_8$ bundle over the Calabi–Yau threefold which is elliptically fibred over $\mathbb{F}_1$, with $\eta = 6c_1(\mathbb{F}_1)$ for each of the two E_8 bundles. The Hodge numbers of the Calabi–Yau fourfold are $h_{11} = 4, h_{31} = 2916, h_{21} = 0, h_{22} = 11724$ and the Euler number is $\chi = 17568$. The cohomology class of the fivebranes is then given by equation 3.9 to be $c_2(\mathbb{F}_1)+91c_1^2(\mathbb{F}_1)$ (hence they all wrap the elliptic fibre), and when integrated over B_2 gives $n_5 = 732 = n_3 = \frac{\chi}{24}$, as expected. Also, using equations 3.7 and 3.12 to count the moduli of the bundles, we find that each E_8 has $I = 1336$, so that $1 + h_{21}(Z) + I_1 + I_2 + h_{21}(X) = 1 + 243 + 1336 + 1336 + 0 = 2916 = h_{31}(X)$, as expected.

5.2 Example 2

Consider now the situation when the second E_8 bundle has $\eta = 6c_1(\mathbb{F}_1) - C_0$ instead. Then the cohomology class of the fivebranes is $C_0\sigma + c_2(\mathbb{F}_1) + 91c_1^2(\mathbb{F}_1) + 15C_0^2 - 45C_0.c_1(\mathbb{F}_1)$. We interpret the first term as a fivebrane wrapping the zero section of $\mathbb{F}_1$ (in what follows, we will use the same notation for the cohomology class of a divisor in the base, and the corresponding curve in the base, and let the context distinguish between the two), and the rest as fivebranes wrapping the elliptic fibre of Z. Integrating over $\mathbb{F}_1$, we find that there are $n_5^{\mathcal{E}} = 672$ fivebranes wrapping the elliptic fibre. On the F-theory side, the new Calabi–Yau fourfold is determined by the change in $\eta(V_2)$ — we find that the only way to achieve this is to blowup $\mathbb{F}_{100}$ by adding the ray $(0, 1, -1)$. Under the natural projection of the base to $\mathbb{F}_1$, this new ray projects to C_0. Furthermore, this ray subdivides the two dimensional cone generated by $(0, 1, 0)$ and $(0, 0, -1)$, and therefore corresponds to a blowup of a curve into a rational surface, and not a blowup of a point into a $\mathbb{P}^2$. Hence we conclude that we are really blowing up the curve C_0 into a rational surface. It follows that the extremal transition from the original Calabi–Yau fourfold to the new Calabi–Yau fourfold is indeed described by replacing a curve in the original fourfold by a dP_8 fibred over it[5].

The Hodge numbers of this fourfold are $h_{11} = 5, h_{31} = 2675, h_{21} = 0, h_{22} = 10764$ and the Euler number is $\chi = 16128$. From the tadpole anomaly cancellation condition 3.8 we expect $n_3 = \frac{\chi}{24} = 672$ threebranes, which is precisely $n_5^{\mathcal{E}}$, the number of fivebranes wrapping the elliptic fibre. We see therefore that the fourfold distinguishes between fivebranes wrapping the elliptic fibre and fivebranes wrapping curves in the base. The first contribute to the number of threebranes, and hence to the Euler number of the fourfold, while the second are seen as blowups in the base B_3 and hence contribute to h_{11}. Note that h_{11} agrees with equation 3.16, since blowing up $\mathbb{F}_{100}$ once increases its h_{11} to 4, and hence increases $h_{11}(X)$ to 5. Finally, using equations 3.7 and 3.12 to count the moduli of the bundles, we find that the second E_8 now has $I = 1095$, so that $1 + h_{21}(Z) + I_1 + I_2 + h_{21}(X) = 1 + 243 + 1336 + 1095 + 0 = 2675 = h_{31}(X)$, as expected.

5.3 Example 3

Now, in the above example, instead of wrapping the fivebrane around C_0, we can choose to wrap it around say, f, by choosing the second E_8 bundle to have $\eta = 6c_1(\mathbb{F}_1) - f$. The cohomology class of the fivebranes is then $f\sigma + c_2(\mathbb{F}_1) + 91c_1^2(\mathbb{F}_1) - 45f.c_1(\mathbb{F}_1)$. The first term then corresponds to a fivebrane wrapping the fibre of $\mathbb{F}_1$, while the rest describe fivebranes wrapping the elliptic fibre of Z. Integrating over $\mathbb{F}_1$, we find that there are $n_5^{\mathcal{E}} = 642$ fivebranes wrapping the elliptic fibre. On the F-theory side, the new Calabi–Yau fourfold is determined by the change in $\eta(V_2)$ — which we achieve by blowing up $\mathbb{F}_{100}$ by adding the ray $(-1, 0, -1)$. Under the natural projection of the base to $\mathbb{F}_1$, this new ray projects to f. Furthermore, this ray subdivides the two dimensional cone generated by $(-1, 0, 0)$ and $(0, 0, -1)$, and therefore corresponds to a blowup of a curve into a rational surface, and not a blowup of a point into a $\mathbb{P}^2$. Again, we conclude that we are really blowing up the curve f

[5]I am grateful to Albrecht Klemm for explaining this point.

into a rational surface. It follows also that the extremal transition from the original Calabi–Yau fourfold to the new Calabi–Yau fourfold is again described by replacing a curve in the original fourfold by a dP_8 fibred over it.

The Hodge numbers of this fourfold are $h_{11} = 5, h_{31} = 2555, h_{21} = 0, h_{22} = 10284$ and the Euler number is $\chi = 15408$. From the tadpole anomaly cancellation condition 3.8 we expect $n_3 = \frac{\chi}{24} = 642$ threebranes, which is precisely $n_5^{\mathcal{E}}$, the number of fivebranes wrapping the elliptic fibre. We see again that the fourfold distinguishes between fivebranes wrapping the elliptic fibre and fivebranes wrapping curves in the base. As in the previous example, the first contribute to the number of threebranes, and hence the Euler number of the fourfold, while the second are seen as blowups in the base B_3 and hence contribute to h_{11}.

Finally, using equations 3.7 and 3.12 to count the moduli of the bundles, we find that the second E_8 now has $I = 974$, so that $1 + h_{21}(Z) + I_1 + I_2 + h_{21}(X) = 1 + 243 + 1336 + 974 + 0 = 2554$, which is one less than $h_{31}(X)$. The missing modulus must be interpreted as a deformation of the curve f that is being wrapped. Let us verify that such is indeed the case. We note that the heterotic base $\mathbb{F}_1$ can be viewed as a blowup (by a $\mathbb{P}^1$) of $\mathbb{P}^2$ at a point p. This blowup is in fact a -1 curve, and is precisely the zero section C_0. Note that since C_0 has negative self-intersection, it cannot be deformed — thus there are no additional complex structure moduli when a fivebrane wraps C_0, which explains the matching of the bundle moduli and $h_{31}(X)$ in example *5.2*. However, the curve in the class f can be deformed (since $f^2 = 0$). In fact, it is a $\mathbb{P}^1$ curve in $\mathbb{P}^2$ which passes through the point p which is being blownup, since $f.C_0 = 1$. Now, a general $\mathbb{P}^1$ curve in $\mathbb{P}^2$ can be described by the equation $ax + by + cz = 0$, where x, y and z are homogeneous coordinates on $\mathbb{P}^2$. This has two deformation parameters, since the equation for the curve is invariant under a rescaling of x, y and z. However, requiring the curve to pass through a given point reduces the number of complex deformations by one and so we have only one parameter. We claim that this parameter is precisely the missing modulus in $h_{31}(X)$. This also implies that the second of equations 3.10 must be modified to include a term corresponding to the deformation moduli of the curves being wrapped.

5.4 Example 4

Let us consider now the situation when the class of the fivebranes wrapping curves in the base is $C_0 + f = C_\infty$, by setting, say, $\eta(V_2) = 6c_1(\mathbb{F}_1) - C_0 - f$. The cohomology class of the fivebranes is now $(C_0 + f)\sigma + c_2(\mathbb{F}_1) + 91c_1^2(\mathbb{F}_1) + 15(C_0 + f)^2 - 45(C_0 + f).c_1(\mathbb{F}_1)$. Integrating over $\mathbb{F}_1$, we obtain $n_5^{\mathcal{E}} = 612$ fivebranes wrapping the elliptic fibre. However, on the F-theory side, we now have two possibilities: we can either blowup once over C_∞ by adding the ray $(0, -1, -1)$, or we can blow up once over C_0 and once over f by adding the rays $(0, 1, -1)$ and $(-1, 0, -1)$. We interpret the first case as corresponding to a fivebrane wrapping C_∞, and the second as corresponding to two fivebranes: one wrapping C_0, and the other wrapping f. In both instances, we are subdividing two dimensional cones, so that we are blowing up curves into ruled surfaces.

However, the Hodge numbers of the corresponding Calabi–Yau fourfolds are different. The first fourfold has Hodge numbers $h_{11} = 5, h_{31} = 2435, h_{21} = 0, h_{22} = 9804$ and Euler number $\chi = 14688$, while the second has $h_{11} = 6, h_{31} = 2434, h_{21} = 0, h_{22} = 9804$ and Euler number $\chi = 14688$. Note, however, that the bundle data (the η_i) are identical, and the class of the fivebranes is also identical. Furthermore, the tadpole anomaly cancellation condition 3.8 predicts $n_3 = \frac{\chi}{24} = 612$ threebranes, in both cases, which is precisely $n_5^{\mathcal{E}}$, the number of fivebranes wrapping the elliptic fibre. The only difference is the specific curve in B_2 that is being wrapped by the fivebranes - in one case we have a single fivebrane wrapping the infinity section leading to one blowup in the base of the fourfold dual, while in the other, we have two fivebranes, one wrapping the fibre, and the other wrapping the zero section of $\mathbb{F}_1$, leading to two blowups in the base of the dual fourfold. Since these two curves lie in the same class, it is conceivable that there is a degeneration that takes one curve to the other. If we were able to follow this degeneration, then we would obtain a heterotic version of the extremal transition on the F-theory side, where we replace one dP_8 fibred over C_∞ by two dP_8's, one fibred over C_0, and the other over f.

Finally, using equations 3.7 and 3.12 to count the moduli of the bundles, we find that the second E_8 now has $I = 853$, so that $1 + h_{21}(Z) + I_1 + I_2 + h_{21}(X) = 1 + 243 + 1336 + 853 + 0 = 2433$, which is two less than $h_{31}(X)$ for the first case and one less than $h_{31}(X)$ for the second. Recall that the second case has one fivebrane wrapping the zero section C_0, and one wrapping f. From our previous example, we know that the curve f has one deformation parameter, while C_0 has none, which accounts for the discrepancy in this case. The two missing moduli in the first case, when the fivebrane wraps C_∞, can also be interpreted as deformation parameters of the curve that is being wrapped. To see this, recall our discussion from example *5.3*, where we regard $\mathbb{F}_1$ as a blowup of $\mathbb{P}^2$ at a point p, with the exceptional $\mathbb{P}^1$ identified with C_0. Now, $C_\infty^2 = 1$, so we see that it can indeed be deformed. Furthermore, $C_\infty . C_0 = 0$, so we see that it is in fact a curve in the $\mathbb{P}^2$ that does not pass through p. Such a curve is then a generic curve in $\mathbb{P}^2$ with two moduli, as discussed above. These are precisely the additional moduli needed to account for the observed discrepancy in $h_{31}(X)$. Finally, note that the degeneration of C_∞ into two curves is easy to describe in this picture. When the curve C_∞ is deformed so as to pass through the point p in $\mathbb{P}^2$, it splits up into two curves, one corresponding to f, and the other to the exceptional divisor, which is precisely C_0. In the process, we lose exactly one deformation modulus, so $h_{31}(X)$ drops by one. Also, since one curve (C_∞) now splits in two (C_0 and f), we gain one Kähler modulus, so $h_{11}(X)$ increases by one. Thus we have described the topology changing extremal transition from the first Calabi–Yau fourfold to the second by a degeneration of curves on the heterotic side. Note, however, that this discussion has been purely classical. We would expect

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this picture to be modified by quantum corrections, in particular, the effects of the superpotential.

More generally, if the heterotic base is $\mathbb{F}_m$, then $C_\infty = C_0 + mf$, and each fivebrane in this class corresponds either to a single blowup (over C_∞) of the Calabi–Yau fourfold base, or $m+1$ blowups, m over f and one over C_0. The choice of the curves being wrapped will then give fourfolds with different Hodge numbers but the same Euler number, since the number of fivebranes wrapping the elliptic fibre of Z, and hence the number of threebranes, is the same. Degenerations from one such curve in $\mathbb{F}_m$ to the other could then be used to follow the topology changing extremal transition from one Calabi–Yau fourfold to the other. A detailed description of these transitions is currently under investigation. We hope to report on this in the future.

6. Discussion

In this paper, we have studied F-theory compactifications on Calabi–Yau fourfolds dual to heterotic compactifications on Calabi–Yau threefolds with some choice of vector bundle. We have shown how toric data (in particular, the polyhedron ∇) for the Calabi–Yau fourfold encode the information about the heterotic dual, in particular, the η class of the heterotic vector bundle. We have found that reflexivity of the fourfold polyhedron imposes a restriction on η, which is related to the stability of the corresponding vector bundle. We have also discussed the construction of Calabi–Yau fourfolds corresponding to three generation models. We note that this construction requires specifying the value of the four flux in addition to the Calabi–Yau fourfold. While the four flux is, in general, independent of the Calabi–Yau fourfold, we have seen that requiring the number of threebranes to be non-negative puts bounds on its value. Thus, not all Calabi–Yau fourfolds can correspond to three generation models.

We have also explored the issue of heterotic fivebranes wrapping curves in the base. We find that these correspond to blowups in the base of the Calabi–Yau fourfold. We have also seen that if the cohomology class of the fivebranes corresponds to a reducible divisor in the heterotic base, then there is an ambiguity in choosing the curves that are being wrapped. This corresponds on the F-theory side to different numbers of blowups, and thus topologically different fourfolds with different Hodge numbers (but same Euler number), and the extremal transitions between these fourfolds can be described (classically) in terms of degenerations of curves in the base of the heterotic threefold.

Thus, we see that, as in the six dimensional situation [5, 7], toric geometry provides a natural arena for discussing F-theory/heterotic duality in four dimensions. The map between toric data and vector bundle data presented in this paper allows us to construct a very rich class of dual pairs. There are, however, many issues in heterotic/F-theory duality which we have not addressed at all in this paper. We have, for instance, not attempted to identify the origin of the matter (in particular, the chiral matter) in the fourfold. Also, we do not have an understanding of the effects of quantum corrections on the proposed duality. Further, we have not studied models with non-perturbative (from the heterotic perspective) gauge groups, nor have we addressed the question of mirror symmetry in fourfolds and its heterotic interpretation. Clearly, we have merely scratched the surface of this subject, and there are many interesting issues to be examined.

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